

Application of Kähler manifold to signal processing and Bayesian inference

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Abstract. We review the information geometry of linear systems and its application to Bayesian inference, and the simplification available in the Kähler manifold case. We find conditions for the information geometry of linear systems to be Kähler, and the relation of the Kähler potential to information geometric quantities such as α -divergence, information distance and the dual α -connection structure. The Kähler structure simplifies the calculation of the metric tensor, connection, Ricci tensor and scalar curvature, and the α -generalization of the geometric objects. The Laplace–Beltrami operator is also simplified in the Kähler geometry. One of the goals in information geometry is the construction of Bayesian priors outperforming the Jeffreys prior, which we use to demonstrate the utility of the Kähler structure.

Keywords: information geometry, Kähler manifold, signal processing, Bayesian inference, Komaki prior, ARFIMA model

INTRODUCTION

Kähler manifolds are important in differential geometry, with applications in several different fields such as supersymmetric gauge theory and superstring theory in theoretical physics, and in our interest, information geometry. After Barndorff-Nielsen and Jupp found the connection between statistics and symplectic geometry [5], Barbaresco introduced Kähler manifolds into information geometry [2] and suggested generalized complex manifolds for information geometry [3, 4]. Symplectic and Kähler structures of divergence functions are also revealed [13]. Recently, Choi and Mullhaupt [6] proved the mathematical correspondence between Kähler manifolds and the information geometry of linear systems. Moreover, the implication of the Kähler manifold to Bayesian inference for linear systems is also reported [7].

Kählerian information geometry has several advantages in describing the information geometry of linear systems [6]. First of all, geometric tensor calculation is simplified. Additionally, the α -generalization of the tensors is more straightforward because the Riemann tensor is α -linear on the complex manifold. Moreover, searching for the superharmonic priors suggested by Komaki [8] is more efficient because the Laplace–Beltrami operator in the Kähler geometry is much simpler. This simplicity leads to a systematic and generic algorithm for the geometric shrinkage priors [7].

In this paper, we give a review on the recent developments in the applications of the Kähler manifold to information geometry, in particular, the implications to signal processing and Bayesian inference. First, we provide the brief fundamentals of the Kähler manifold, and the Kählerian description of linear systems is introduced. After then, we present an application to Bayesian inference.

REVIEW ON KÄHLER MANIFOLD

We review the fundamentals of the Kähler manifold in this section. Let us start with construction of a complex manifold. To extend a real manifold to a complex manifold, the concept of complexification is necessary. Any functions and vectors can be complexified. The complexified coordinate system $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{C}^n$ of a complex manifold M is given by

$$\xi = \theta + i\zeta$$

where θ and ζ are the real coordinate systems of a real manifold N of dimension n . The complex manifold M is the complexification of the manifold N , denoted by $N^{\mathbb{C}}$, and it can be considered as a product manifold $N \times N$. From now on, we work on the complex manifold M of $\dim_{\mathbb{C}} M = n$.

On the tangent plane at point p , also denoted by $T_p M$, the basis vectors are given by the real coordinates as

$$\left\{ \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^n}; \frac{\partial}{\partial \zeta^1}, \dots, \frac{\partial}{\partial \zeta^n} \right\}$$

and the cotangent plane $T_p^* M$, which is dual to the tangent plane, is spanned by

$$\{d\theta^1, \dots, d\theta^n; d\zeta^1, \dots, d\zeta^n\}$$

where $d\theta^i$ and $d\zeta^i$ are the one-forms of the manifold. Since the vectors on the tangent space and the one-forms on the cotangent space are dual to each other, the basis vectors and the one-forms should satisfy the following identities:

$$\begin{aligned} \langle d\theta^i, \frac{\partial}{\partial \theta^j} \rangle &= \delta_j^i, \langle d\theta^i, \frac{\partial}{\partial \zeta^j} \rangle = 0 \\ \langle d\zeta^i, \frac{\partial}{\partial \theta^j} \rangle &= 0, \langle d\zeta^i, \frac{\partial}{\partial \zeta^j} \rangle = \delta_j^i \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product and δ_j^i is the Kronecker delta.

It is also possible to describe the manifold with the complexified coordinate system. First of all, let us introduce the following vectors:

$$\frac{\partial}{\partial \xi^i} = \frac{1}{2} \left(\frac{\partial}{\partial \theta^i} - i \frac{\partial}{\partial \zeta^i} \right), \frac{\partial}{\partial \bar{\xi}^i} = \frac{1}{2} \left(\frac{\partial}{\partial \theta^i} + i \frac{\partial}{\partial \zeta^i} \right).$$

The tangent space $T_p M$ is spanned by the basis vectors defined above:

$$\left\{ \frac{\partial}{\partial \xi^1}, \dots, \frac{\partial}{\partial \xi^n}, \frac{\partial}{\partial \bar{\xi}^1}, \dots, \frac{\partial}{\partial \bar{\xi}^n} \right\}$$

and its dual cotangent space $T_p^* M$ is spanned by

$$\{d\xi^1, \dots, d\xi^n, d\bar{\xi}^1, \dots, d\bar{\xi}^n\}$$

such that the vectors and the one-forms in the complexified coordinate system satisfy the similar identities in the case of the real basis vectors and one-forms:

$$\begin{aligned}\langle d\xi^i, \frac{\partial}{\partial \xi_j} \rangle &= \delta_j^i, \langle d\xi^i, \frac{\partial}{\partial \bar{\xi}_j} \rangle = 0 \\ \langle d\bar{\xi}^i, \frac{\partial}{\partial \xi_j} \rangle &= 0, \langle d\bar{\xi}^i, \frac{\partial}{\partial \bar{\xi}_j} \rangle = \delta_j^i.\end{aligned}$$

The manifold has the almost complex structure that is the linear mapping $J_p : T_p M \rightarrow T_p M$ with

$$J_p \frac{\partial}{\partial \xi^i} = i \frac{\partial}{\partial \xi^i}, J_p \frac{\partial}{\partial \bar{\xi}^i} = -i \frac{\partial}{\partial \bar{\xi}^i}$$

and its matrix representation is the following:

$$J_p = \begin{pmatrix} i \mathbb{I}_n & 0 \\ 0 & -i \mathbb{I}_n \end{pmatrix}$$

where \mathbb{I}_n the identity matrix of dimension n .

A Hermitian manifold is defined as a complex manifold equipped with the metric tensor g_p of the following property:

$$g_p(J_p X, J_p Y) = g_p(X, Y)$$

where $X, Y \in T_p M$. First of all, the definition of the Hermitian manifold can be represented in terms of the metric tensor components as follows:

$$g_{ij} = g_{\bar{i}\bar{j}} = 0 \quad (1)$$

where the metric elements with the mixed indices may not vanish. In addition to that, it is always possible to construct a Hermitian manifold from any complex manifold.

One more concept for defining the Kähler manifold is the Kähler form. The Kähler form is defined as

$$\Omega_p = g_p(J_p X, Y)$$

where $X, Y \in T_p M$. It is antisymmetric under the exchange of X and Y : $\Omega_p(X, Y) = -\Omega_p(Y, X)$. It is expressed in terms of the metric tensor components:

$$\Omega = i g_{i\bar{j}} d\xi^i \wedge d\bar{\xi}^j$$

where \wedge is the wedge product.

Now, we are ready for defining the Kähler manifold. The Kähler manifold is defined as the Hermitian manifold with the closed Kähler form. The closed Kähler two-form, $d\Omega = 0$, is written in the metric tensor components

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}, \partial_{\bar{i}} g_{k\bar{j}} = \partial_{\bar{j}} g_{k\bar{i}}. \quad (2)$$

In the metric tensor expression, the geometry is Kähler if and only if the metric tensor satisfies eq. (1) and eq. (2).

One of the most well-known properties in the Kähler geometry is that the metric components on the Kähler manifold is given by the Hessian structure:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K} \quad (3)$$

where \mathcal{K} is the Kähler potential. All the information on the metric tensor is encoded in the Kähler potential. The nontrivial elements of the Levi-Civita connection also can be expressed with the Kähler potential:

$$\Gamma_{ij,\bar{k}} = \partial_i \partial_j \partial_{\bar{k}} \mathcal{K} = (\Gamma_{\bar{i}\bar{j},k})^* \quad (4)$$

and the other elements of the Levi-Civita connection vanish. The connection with this property is called the Hermitian connection.

Another notable fact in the Kähler geometry is that the Ricci tensor is calculated from

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \mathcal{G} \quad (5)$$

where \mathcal{G} is the determinant of the metric tensor. The lengthy calculation for the Riemann curvature tensor can be skipped in the procedure of obtaining the Ricci tensor.

Additionally, the submanifolds of the Kähler manifolds are also Kähler. If a given manifold is the Kähler manifold, every submanifolds are automatically Kähler.

Finally, it is noteworthy that the Laplace–Beltrami operator is represented with

$$\Delta = 2g^{i\bar{j}} \partial_i \partial_{\bar{j}}$$

and it is much simpler than the Laplace–Beltrami operator of a non-Kähler manifold.

KÄHLER GEOMETRY OF SIGNAL PROCESSING

In this section, we cover the Kählerian information geometry for signal processing proposed by Choi and Mullhaupt [6]. A signal filter transforms an input signal x to an output y under the following linear relation:

$$y(w) = h(w; \xi) x(w; \xi)$$

where $h(w; \xi)$ is a transfer function in frequency domain w . The z -transformed transfer function of a causal filter is expressed by

$$h(z; \xi) = \sum_{r=0}^{\infty} h_r(\xi) z^{-r}$$

where h_r is the r -th impulse response function of the linear system. We assume that $h(z; \xi)$ is holomorphic both in ξ and z .

It is well-known by Amari and Nagaoka [1] that the metric tensor is determined for stationary processes by the spectral density function $S(z; \xi) = |h(z; \xi)|^2$. It is also possible to write down the metric tensor in terms of the transfer function on the complexified

manifold. The metric components can be represented with η_r , the coefficient of z^{-r} in the logarithmic transfer function (log-transfer function),

$$g_{ij} = \partial_i \eta_0 \partial_j \eta_0 \quad (6)$$

$$g_{i\bar{j}} = \sum_{r=0}^{\infty} \partial_i \eta_r \partial_{\bar{j}} \bar{\eta}_r \quad (7)$$

where $g_{i\bar{j}}$ and g_{ij} are the complex conjugates of g_{ij} and $g_{i\bar{j}}$, respectively. It is straightforward that $\eta_0 = \log h_0$.

Choi and Mullhaupt [6] proved that the information geometry of stationary and minimum phase linear systems is Kähler. They also provided the conditions on the transfer function of a linear system where the information geometry is the Kähler manifold in which the Hermitian conditions, eq. (1), are explicitly shown at the induced metric level. In this paper, we confine ourselves to the Kähler manifold with explicit Hermitian metric properties, eq. (1). In the case of a causal filter, the condition for Kähler manifold is as follows.

Theorem 1. *Given a holomorphic transfer function, the information geometry of a signal filter is the Kähler manifold if and only if h_0 is a constant in ξ .*

Proof. If h_0 is a constant, the metric tensor expressions, eq.(6) and eq. (7), are given by

$$g_{ij} = g_{i\bar{j}} = 0, g_{i\bar{j}} = \sum_{r=1}^{\infty} \partial_i \eta_r \partial_{\bar{j}} \bar{\eta}_r$$

i.e. the manifold is Hermitian. Additionally, it is easy to check that the Kähler form is closed.

If the geometry is Kähler, the manifold is Hermitian where $g_{ij} = g_{i\bar{j}} = 0$ for all i and j . From this Hermitian condition, it is obvious that h_0 is constant in ξ . \square

For the Kählerian linear systems, the Kähler potential is the square of the Hardy norm of the log-transfer function on the unit disk \mathbb{D} [6]:

$$\mathcal{K} = \frac{1}{2\pi i} \oint_{|z|=1} |\log h(z; \xi)|^2 \frac{dz}{z} = \|\log h(z; \xi)\|_{H^2}^2 \quad (8)$$

and the Kähler potential is also related to the 0-divergence. It is identical to the 0-divergence for the unilateral transfer function. It is a constant in α of α -divergence.

According to the literature [6], the benefits of the Kählerian information geometry are the followings. First of all, the calculation of the geometric tensors and Levi-Civita connection is simplified by the Kähler structure and the expressions for the geometric objects are given by eq. (3), eq. (4), and eq. (5). Additionally, the α -generalization of the tensors are still α -linear. Finally, it is easier to find superharmonic priors on the manifold because the Laplace–Beltrami operator on the Kähler manifold is in the simpler form.

We give an example: one of the most interesting linear systems is the fractionally integrated autoregressive moving average (ARFIMA) model. For the ARFIMA(p, d, q)

model of $\xi = (d, \lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q)$, the transfer function rescaled by the gain is given by

$$h(z; \xi) = \frac{(1 - \mu_1 z^{-1})(1 - \mu_2 z^{-1}) \cdots (1 - \mu_q z^{-1})}{(1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) \cdots (1 - \lambda_p z^{-1})} (1 - z^{-1})^d$$

where λ_i is a pole from the AR part, μ_i is a root from the MA part, and d is a differencing parameter. The poles and the roots are expected to be on the unit disk. By Theorem 1, it is clear that the information geometry of the ARFIMA model is Kähler. The Kähler potential of the ARFIMA model, also found in the literature [7], is calculated from eq. (8) as

$$\mathcal{K} = \sum_{k=1}^{\infty} \left| \frac{d + (\mu_1^k + \cdots + \mu_q^k) - (\lambda_1^k + \cdots + \lambda_p^k)}{k} \right|^2 \quad (9)$$

and it is bounded above by $(d + p + q)^2 \frac{\pi^2}{6}$. The metric tensor, derived from eq. (3), is represented by

$$g_{i\bar{j}} = \begin{pmatrix} \frac{\pi^2}{6} & \frac{1}{\lambda_j} \log(1 - \bar{\lambda}_j) & -\frac{1}{\bar{\mu}_j} \log(1 - \bar{\mu}_j) \\ \frac{1}{\lambda_i} \log(1 - \lambda_i) & \frac{1}{1 - \lambda_i \bar{\lambda}_j} & -\frac{1}{1 - \lambda_i \bar{\mu}_j} \\ -\frac{1}{\mu_i} \log(1 - \mu_i) & -\frac{1}{1 - \mu_i \bar{\lambda}_j} & \frac{1}{1 - \mu_i \bar{\mu}_j} \end{pmatrix}$$

where the first column and the first row are for the direction of the fractional differencing parameter d . It is easy to find the metric tensor for the pure ARMA model as a submanifold of the ARFIMA geometry.

The non-trivial connection elements are also found from eq. (4) and it is noteworthy that the connection components with the differencing parameter direction at any index of the first two indices are all vanishing. The Ricci tensor components are also calculated by eq. (5) and it is also vanishing along the d -direction. The non-vanishing Ricci tensor components are only from the pure ARMA directions with the correction term from the mixing between the pure ARMA piece and the fractionally integrated part:

$$R_{i\bar{j}} = R_{i\bar{j}}^{ARMA} + R_{i\bar{j}}^{ARMA-FI}$$

where i and j are not along the d -direction.

GEOMETRIC SHRINKAGE PRIORS OF KÄHLERIAN FILTERS

First, we review the superharmonic priors proposed by Komaki [8]. The difference in risk function between two Bayesian predictive densities from the Jeffreys prior π_J and the superharmonic prior π_I with respect to the density $p(y|\xi)$ is given by

$$\begin{aligned} & \mathbb{E}[D_{KL}(p(y|\xi) || p_{\pi_J}(y|x^{(N)})) | \xi] - \mathbb{E}[D_{KL}(p(y|\xi) || p_{\pi_I}(y|x^{(N)})) | \xi] \\ &= \frac{1}{2N^2} g^{ij} \partial_i \log\left(\frac{\pi_I}{\pi_J}\right) \partial_j \log\left(\frac{\pi_I}{\pi_J}\right) - \frac{1}{N^2} \frac{\pi_J}{\pi_I} \Delta\left(\frac{\pi_I}{\pi_J}\right) + o(N^{-2}) \end{aligned}$$

where N is the size of samples x . If a positive prior function $\psi = \pi_I/\pi_J$ is superharmonic, the risk function of the Bayesian predictive density p_{π_I} is decreased with respect to that of p_{π_J} , the predictive density from the Jeffreys prior. Comparing with p_{π_I} , p_{π_I} is closer to $p(y|\xi)$ in the Kullback-Leibler divergence. Superharmonic priors for several probability distributions and linear systems have been found [8, 11, 12, 6, 7].

A difficulty in Komaki's idea is that it is non-trivial to test the superharmonicity for a prior function ψ of a general statistical model or a linear system with high dimensionality. Although a superharmonic prior for the AR model in an arbitrary dimension was found by Tanaka [12], no superharmonic priors for the ARMA models and the ARFIMA models were found. Moreover, any systematic algorithms for finding the Komaki priors were not known.

Recently, a generic algorithm for the shrinkage priors of linear systems is introduced when the information geometry of the model is Kähler [7]. Superharmonic priors for more general time series models and signal filters are efficiently constructed by the algorithm. The following theorem is useful to find the superharmonic prior functions.

Theorem 2. *On a Kähler manifold, a positive function $\psi = \Psi(u^* - \kappa(\xi, \bar{\xi}))$ is a superharmonic prior function if $\kappa(\xi, \bar{\xi})$ is subharmonic (or harmonic), bounded above by u^* , and Ψ is concave decreasing: $\Psi'(\tau) > 0$, $\Psi''(\tau) \leq 0$ (or $\Psi'(\tau) > 0$, $\Psi''(\tau) < 0$).*

Proof. The proof is given in the literature [7]. □

If we find a positive subharmonic or harmonic function, we apply Theorem 2 to obtain a superharmonic function and exploit the superharmonic function as a shrinkage prior function for prediction as Komaki [8] suggested.

Fortunately, several choices for Ψ and κ are already known [7]. The candidates for Ψ are the followings:

$$\begin{aligned}\Psi_1(\tau) &= \tau^a \\ \Psi_2(\tau) &= \log(1 + \tau^a)\end{aligned}$$

where τ is positive and $0 < a \leq 1$ for subharmonic κ (or $0 < a < 1$ for harmonic κ). Moreover, the ansätze for κ are found as

$$\begin{aligned}\kappa_1 &= \mathcal{K} \\ \kappa_2 &= \sum_{r=0}^{\infty} a_r |h_r(\xi)|^2 \\ \kappa_3 &= \sum_{i=1}^n b_i |\xi^i|^2\end{aligned}$$

where a_r and b_i are positive real numbers. In particular, κ_1 is the Kähler potential which is intrinsic on the Kähler manifold. By combining κ and Ψ , it is easy to construct geometric shrinkage priors like

$$\begin{aligned}\psi_1 &= (u^* - \mathcal{K})^a \\ \psi_2 &= \log(1 + (u^* - \mathcal{K})^a)\end{aligned}$$

which outperform the Jeffreys prior in the viewpoint of information theory.

CONCLUSION

We reviewed information geometric applications of Kähler manifolds to linear systems and Bayesian inference, and exposed that the simpler Laplace–Beltrami operator, one of the advantages in the Kählerian approach, is applicable to Bayesian inference: finding superharmonic priors on the Kähler manifold is straightforward, as we have shown for linear systems, in particular, the ARFIMA models.

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